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THE LYAPUNOV FUNCTIONS IN ESTIMATES OF A DIMENSION OF DYNAMICAL SYSTEM ATTRACTORS

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It is well known that the numerical characteristics of harmonic oscillations are the following: an amplitude, a period, a frequency and for a periodic oscillation this is a period. Numerous investigations show that more complicated oscillations have also numeric characteristics, namely, the dimensions of attractors, which correspond to the ensembles of such oscillations.

The theory of topological dimensionality [1, 2], developed in the first half of the last century, is not suitable for a scale of dimensional characteristics of attractors to be given. The cause is that a topological dimension can take on integer values only and a scale of dimensional characteristics, obtained in such a way, turns out to be poor.

In studying the attractors the Hausdorff dimension of sets is more suitable. This dimensional characteristic can have any nonnegative values and for Euclidean space usual objects like a smooth curve, a surface, and a countable set of points it coincide with the topological dimension.

Often, the attractors that have a nonintegral Hausdorff dimension are called strange attractors. At present by means of computer experiments many of the strange attractors are found, which describe complicated oscillations occuring in nature and technics [3—6].

Let us now define the Hausdorff dimension of sets.

We consider a compact metric set X with metric ρ, a subset $E \subset X$, and the numbers $d \geq 0$, $\varepsilon > 0$.

Cover E by the balls of radii $r_j < \varepsilon$ and denote

$$\mu_H(E, d, \varepsilon) = \inf \sum_j r_j^d,$$

where the infimum is taken over all such ε-covers of E.

Obviously, $\mu_H(E, d, \varepsilon)$ does not decrease with decreasing ε. Therefore there exists a limit (which may also be infinite) such that

$$\mu_H(E, d) = \lim_{\varepsilon \to 0} \mu_H(E, d, \varepsilon).$$

Definition 1. The function $\mu_H(\cdot, d)$ is called a *d-measure of Hausdorff.*

For a fixed d the function $\mu_H(E, d)$ has all the properties of an outer measure on X.

For a fixed set E the function $\mu_H(E, \cdot)$ has the following property. There exists $d_{cr} \in [0, \infty]$ such that

$$\mu_H(E, d) = \infty, \quad \forall\, d < d_{kp},$$
$$\mu_H(E, d) = 0, \quad \forall\, d > d_{cr}.$$

If $X \subset \mathbb{R}^n$, then $d_{cr} \leq n$. Here $\mathbb{R}^n$ is an Euclidean n-dimensional space.

Put

$$\dim_H E = d_{cr} = \inf \{d| \ \mu_H(E,d) = 0\}.$$

Definition 2. The quantity $\dim_H E$ is called the *Hausdorff dimension* of set E.

Example 1. Consider a Cantor set

$$E = \bigcap_{j=0}^{\infty} E_j,$$

where $E_0 = [0,1]$ and E_j consist of 2^j segments of length 3^{-j}, obtained from segments entering E_{j-1} by deleting the midpoints of the open intervals of lengths 3^{-j} (Fig. 1).

Fig. 1.

¿From the classic theory of topological dimension it is well known that $\dim_T E = 0$.

In addition from the definition of the Hausdorff dimension it follows that $\mu_H(E,d) = 1$ for $d = \log 2/\log 3 = 0,6309...$ and, consequently,

$$\dim_H E = \frac{\log 2}{\log 3}.$$

A topological dimension is invariant to homeomorphismes. The Hausdorff dimension is invariant to diffeomorphismes, and the nonintegral Hausdorff dimension is not invariant to homeomorphismes [2].

When considering the attractors of dynamic systems in the phase spaces the smooth coordinate changes are often used. Therefore in these cases it is sufficient that there exists the invariance with respect to diffeomorphismes.

It is well known that $\dim_T E \leq \dim_H E$. In the case of the Cantor set E this inequality is strict.

Let us give now two equivalent definitions of fractal dimension.

Denote by $\mathcal{N}_\varepsilon(E)$ a minimal number of the balls of radius ε to be necessary to cover a set $E \subset X$.

Take the numbers $d \geq 0$, $\varepsilon > 0$ and put

$$\mu_F(E,d,\varepsilon) = \mathcal{N}_\varepsilon(E)\varepsilon^d,$$
$$\mu_F(E,d) = \overline{\lim_{\varepsilon \to 0}} \mu_F(E,d,\varepsilon).$$

Definition 3. A value

$$\dim_F E = \inf\{d| \ \mu_F(E,d) = 0\}.$$

is called *a fractal dimension* of set E.

Note that a scheme of the introduction of this definition is similar to that given for the Hausdorff dimension. But here the coverings are performed by the balls of the same radius ε.

Definition 4. *A fractal dimension* of set E is a quantity

$$\dim_F E = \overline{\lim_{\varepsilon \to 0}} \frac{\log \mathcal{N}_\varepsilon(E)}{\log(1/\varepsilon)}.$$

It is easy to see that

$$\dim_H E \leq \dim_F E.$$

Example 2. For $X = [0,1]$ and $E = \{0, 1, 2^{-1}, 3^{-1}, \ldots\}$

$$\dim_H E = 0, \quad \dim_F E = \frac{1}{2}.$$

Some extension of schemes of an introduction of the Hausdorff and fractal measures and dimensions and the definitions of different metrical dimensional characteristics are given in [7].

We can see [8–13] that the upper estimate of the Hausdorff and fractal dimensions of invariant sets is the Lyapunov dimension. Let us give its definition.

Consider some continuously differentiable mapping F of a set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Denote by $T_x F$ the Jacobian matrix of the mapping F at the point x. From the continuous differentiability of F it follows that

$$F(x+h) - F(x) = (T_x F)h + o(h).$$

We assume further that a set $K \subset U$ is invariant with respect to the mapping $F: \ F(K) = K$.

Consider singular numbers of $n \times n$-matrix A

$$\alpha_1(A) \geq \ldots \geq \alpha_n(A).$$

Recall that *a singular number* of matrix A is a square root from an eigenvalue of matrix A^*A. Here $*$ denotes the operation of transposition (in the real case) or that of the Hermitean conjugacy.

Definition 5. *A local Lyapunov dimension of mapping F at the point $x \in K$* is a number

$$\dim_L(F, x) = j + s,$$

where j is the greatest integer number from the segment $[1, n]$, for which

$$\alpha_1(T_x F) \ldots \alpha_j(T_x F) \geq 1,$$

and the number $s \in [0,1]$ is such that

$$\alpha_1(T_x F) \ldots \alpha_j(T_x F) \alpha_{j+1}(T_x F)^s = 1.$$

By definition we have $\dim_L(F,x)=0$ in the case $\alpha_1(T_xF)<1$ and $\dim_L(F,x)=n$ in the case

$$\alpha_1(T_xF)\ldots\alpha_n(T_xF)\geq 1.$$

Definition 6. *The Lyapunov dimension of the mapping F of set K* is a number

$$\dim_L(F,K)=\sup_K \dim_L(F,x).$$

Definition 7. *A local Lyapunov dimension of a sequence of mappings F^i at the point $x\in K$* is a number

$$\dim_L x=\overline{\lim_{i\to+\infty}}\,\dim_L(F^i,x).$$

Definition 8. *The Lyapunov dimension of a sequence of mappings F^i of set K* is a number

$$\dim_L K=\sup_K \dim_L x.$$

For mappings F_t depending on a parameter $t\in\mathbb{R}^1$ it makes possible to introduce the definitions similar to Definitions 7 and 8.

Definition 9. *A local Lyapunov dimension of a mapping F_t at the point $x\in K$* is a number

$$\dim_L x=\overline{\lim_{t\to+\infty}}\,\dim_L(F_t,x).$$

Definition 10. *The Lyapunov dimension of a mapping F_t of set K* is a number

$$\dim_L K=\sup_K \dim_L x.$$

Note once more that an important property of the Lyapunov dimension is an inequality [11–13] $\dim_F K\leq\dim_L K$.

Thus

$$\dim_T K\leq\dim_H K\leq\dim_F K\leq\dim_L K. \tag{1}$$

Note also that the Lyapunov dimension can be a characteristic of the inner instability of a dynamic system, defined on the invariant set K and generated by the family of mappings F^i or F_t.

The Lyapunov dimension is invariant with respect to the linear orthogonal operators. Sometimes this implies that in studying the concrete dynamic systems some "good" (or the best, in a certain sense) basis must be found.

It is necessary to note that if we pass from topological dimensions to the Hausdorff ones, we stretch a topology since the Hausdorff dimension is not invariant with respect to homeomorphismes. The introduction of the Lyapunov dimension is the next step in the pragmatic approach to the study of dynamic systems: the Lyapunov dimension is not a dimensional characteristic in the classical sense. But it makes possible, firstly, to estimate effectively from above the Hausdorff and fractal dimensions. Secondly, it is a characteristic of instability of dynamic systems and, thirdly, it will be "convenient" for the investigation of a motion stability by the classical theory methods.

Let us show this, using the Lyapunov functions in the estimates of the Lyapunov dimension. The idea to introduce the Lyapunov functions in the estimates of dimensional characteristics was formulated first in the work [14] and was developed further in [15–20, 9, 10]. Here we follow mainly these ideas.

Consider a continuously differentiable function $p(x) : U \to \mathbb{R}^1$, which satisfies the following condition

$$p_1 \leq p(x) \leq p_2, \quad \forall\, x \in U.$$

Here p_1 and p_2 are some positive numbers.

Let j be an integer number from the segment $[1, n]$, $s \in [0, 1)$, $d = j + s$, A be an $n \times n$-matrix,

$$\omega_d(A) = \alpha_1(A) \ldots \alpha_j(A) \alpha_{j+1}(A)^s,$$

$$\omega(d, K, p, F) = \sup_K \omega_d \left(\frac{p(F(x))}{p(x)} T_x F \right).$$

T h e o r e m 1. *If $F(K) = K$ and the following inequality holds*

$$\omega(d, K, p, F) < 1, \tag{2}$$

then for sufficiently large natural i we have

$$\dim_L(F^i, K) \leq d. \tag{3}$$

P r o o f. For two $n \times n$-matrices A and B the inequality [21]

$$\omega_d(AB) \leq \omega_d(A) \omega_d(B) \tag{4}$$

is well known. Using the relation

$$T_x F^i = \left(T_{F^{i-1}(x)} F \right) \left(T_{F^{i-2}(x)} F \right) \ldots \left(T_x F \right),$$

we obtain

$$T_x F^i = \frac{p(x)}{p(F^i(x))} \left(\frac{p(F^i(x))}{p(F^{i-1}(x))} T_{F^{i-1}(x)} F \right) \ldots \left(\frac{p(F(x))}{p(x)} T_x F \right).$$

This and inequality (4) imply the following estimate

$$\omega_d(T_x F^i) \leq \sup_K \left(\frac{p(x)}{p(F^i(x))} \right)^d \omega(d, K, p, F)^i.$$

¿From this inequality, condition (2), and the definition of the Lyapunov dimension estimate (3) follows.

As it will be shown further, the factors of the form

$$\frac{p(F(x))}{p(x)}$$

in condition (2) play the role of the functions of the Lyapunov type. It can clearly be seen when studying the dynamical systems generated by differential equations.

Consider a system

$$\frac{dx}{dt} = f(t, x), \quad t \in \mathbb{R}^1, \ x \in \mathbb{R}^n, \tag{4}$$

where $f(t,x)$ is a continuously differentiable T-periodic vector function $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t+T, x) = f(t, x)$.

Suppose that a solution $x(t, x_0)$ of system (4) with the initial data $x(0, x_0) = x_0$ are defined on the segment $[0, T]$ and denote by G_T a shift operator along a solution of system (4):

$$G_T q = x(T, q).$$

Let us assume that for a certain set K the following relation holds

$$G_T K = K.$$

Denote by $J(t, x)$ the Jacobian matrix of vector function $f(t, x)$:

$$J(t, x) = \frac{\partial f(t, x)}{\partial x}.$$

Let $\lambda_1(t, x) \geq \ldots \geq \lambda_n(t, x)$ be the eigenvalues of the matrix

$$\frac{1}{2}\,(J(t, x) + J(t, x)^*).$$

Here $*$ denotes an operation of conjugation.

The,orem 2. *Suppose, there exists an integer number $j \in [1, n]$, a number $s \in [0, 1)$, and a continuously differentiable function $v(x)$ such that the following inequality*

$$\sup_{q \in K} \int_0^T (\lambda_1(t, x(t, q)) + \ldots + \lambda_j(t, x(t, x)) + \tag{5}$$
$$+ s\lambda_{j+1}(t, x(t, q)) + \dot{v}(x(t, q)))\, dt < 0$$

is satisfied. Then for sufficiently large natural i the inequality holds

$$\dim_L(G_T^i, K) \leq j + s. \tag{6}$$

In the analysis of dynamical systems in accordance with the direct Lyapunov method it will sometimes be convenient to use a more rough condition rather than inequality (5):

$$\sup_{q\in K}\big(\lambda_1(t,x(t,q))+\ldots+\lambda_j(t,x(t,x))+ \\ +s\lambda_{j+1}(t,x(t,q))+\dot v(x(t,q))\big)<0,\quad \forall\, t\in\mathbb{R}^1. \tag{7}$$

To prove Theorem 2, we need a result to be valid for the linear system

$$\frac{dX}{dt}=A(t)X, \tag{8}$$

where $A(t)$ is a real continuous $n\times n$-matrix, $X(t)$ is an $n\times n$-matrix being a solution of system (8) with the initial condition $X(0)=I$.

Denote by $\lambda_1(t)\geq\ldots\geq\lambda_n(t)$ the eigenvalues of the matrix

$$\frac{1}{2}\left(A(t)+A(t)^*\right)$$

and by $\sigma_1(t)\geq\ldots\geq\sigma_n(t)$ denote the singular numbers of the matrix $X(t)$.

L e m m a 1. *The following inequality holds*

$$\sigma_1(t)\ldots\sigma_j(t)\leq\exp\Big(\int_0^t(\lambda_1(\tau)+\ldots+\lambda_j(\tau))\,d\tau\Big). \tag{9}$$

P r o o f. We can easily see that the numbers $\sigma_k(t+\Delta t)$ are the singular values of the matrix

$$X(t+\Delta t)=(I+A(t)\Delta t+o(\Delta t))X(t).$$

¿From here and from inequality (4) it follows that

$$\sigma_1(t+\Delta t)\ldots\sigma_j(t+\Delta t)\leq\beta_1\ldots\beta_j\sigma_1(t)\ldots\sigma_j(t), \tag{10}$$

where β_k are the singular values of the matrix $I+A(t)\Delta t+o(\Delta t)$.

Recall that β_k^2 is the eigenvalue of the matrix

$$(I+A(t)\Delta t+o(\Delta t))^*(I+A(t)\Delta t+o(\Delta t))= \\ =I+(A(t)^*+A(t))\Delta t+o(\Delta t).$$

Thus, we have $\beta_k^2=1+2\lambda_k(t)\Delta t+o(\Delta t)$ and, consequently,

$$\beta_k=1+\lambda_k(t)\Delta t+o(\Delta t).$$

This and (10) imply that

$$\sigma_1(t+\Delta t)\ldots\sigma_j(t+\Delta t)\leq \\ \Big(1+\Delta t\sum_{k=1}^{j}\lambda_k(t)+o(\Delta t)\Big)\sigma_1(t)\ldots\sigma_j(t). \tag{11}$$

By (11)

$$\frac{d}{dt}\,[\sigma_1(t)\ldots\sigma_j(t)] \le \Big(\sum_{k=1}^{j}\lambda_k(t)\Big)[\sigma_1(t)\ldots\sigma_j(t)]. \tag{12}$$

Integrating differential inequality (12), we obtain inequality (9).

Note that the present proof of (9) is more natural and simple than the proves, given in [9, 10].

P r o o f o f T h e o r e m 2. Denote

$$H(t,q) = \frac{\partial x(t,q)}{\partial q}\,.$$

Substituting the solution $x(t,q)$ into equation (4) and differentiating both sides with respect to q, we obtain

$$\frac{dH(t,q)}{dt} = J(t,x(t,q))H(t,q).$$

By the lemma for the singular values $\sigma_1(t) \ge \ldots \sigma_n(t)$ of the matrix $H(t,q)$ we have

$$\sigma_1\ldots\sigma_k \le \exp\Big(\int_0^t (\lambda_1+\ldots+\lambda_k)\,d\tau\Big)$$

for any $k = 1,\ldots,n$. From here and from the following equality

$$\sigma_1\ldots\sigma_j\sigma_{j+1}^s = (\sigma_1\ldots\sigma_j)^{1-s}(\sigma_1\ldots\sigma_{j+1})^s$$

the estimate

$$\sigma_1\ldots\sigma_j\sigma_{j+1}^s \le \exp\Big(\int_0^t (\lambda_1+\ldots+\lambda_j+s\lambda_{j+1})\,d\tau\Big) \tag{13}$$

follows. Put

$$p(x) = (\exp v(x))^{1/(j+s)}$$

and multiply both sides of inequality (13) by the following expression

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s} = \exp\,[v(x(t,q)-v(q)] = \exp\Big(\int_0^t \dot v(x(\tau,q))\,d\tau\Big).$$

As a result, we obtain the inequality

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s}\sigma_1\ldots\sigma_j\sigma_{j+1}^s \le$$

$$\le \exp\Big(\int_0^t (\lambda_1+\ldots+\lambda_j+s\lambda_{j+1}+\dot v(x,\tau,q))\,d\tau\Big).$$

This implies the following estimate

$$\alpha_1(t,q)\dots\alpha_j(t,q)\alpha_{j+1}(t,q)^s \le \exp\Big(\int\limits_0^t (\lambda_1(\tau,x(\tau,q))+ \\ +\dots+\lambda_j(\tau,x(\tau,q))+s\lambda_{j+1}(\tau,x(\tau,q))+\dot v(x(\tau,q)))d\tau\Big), \tag{14}$$

where $\alpha_k(t,q)$ are the singular values of the matrix

$$\frac{p(x(t,q))}{p(q)}\,H(t,q).$$

¿From estimate (14) amd condition (5) of Theorem 2 it follows that there exists a positive number ε such that the equality

$$\alpha_1(T,q)\dots\alpha_j(T,q)\alpha_{j+1}(T,q)^s \le \exp(-\varepsilon)$$

is valid for all $q\in K$.

Thus, for $F=G_T$ all the conditions of Theorem 1 are satisfied, we have

$$T_qF=T_qG_T=H(T,q),$$

and therefore estimate (6) holds. Theorem 2 is proved.

We now apply Theorems 1 and 2 to concrete dynamic systems, constructing the Lyapunov functions $p(x)$ (in the discrete cases) and $v(x)$ (for dynamic systems of the form (4)).

The publication of results similar to those, formulated in Theorems 1 and 2 [14], causes the construction of the Lyapunov functions for an estimation of dimensions of the invariant sets of dynamic systems [15–20, 9, 10]. Following mainly these works we will demonstrate here the construction of the Lyapunov functions for the well-known dynamic systems with chaotic behavior: for the Hennon mapping and the Lorenz and Rössler systems.

Consider the Hennon mapping $F:\mathbb{R}^2\to\mathbb{R}^2$

$$\begin{aligned}&x\to a+by-x^2,\\&y\to x,\end{aligned} \tag{15}$$

where $a>0$, $b\in(0,1)$ are parameters of mapping.

Having performed the change of variables

$$x\to x,\quad y\to y\sqrt{b}\,,$$

we pass from (15) to a mapping F_1:

$$\begin{aligned}&x\to a+\sqrt{b}\,y-x^2,\\&y\to\sqrt{b}\,x.\end{aligned} \tag{16}$$

Consider now the invariant sets K of mapping (16), $F_1 K = K$, containing the stationary points of this mapping:

$$x_+ = \frac{1}{2}\left[b - 1 + \sqrt{(b-1)^2 + 4a}\,\right],$$

$$x_- = \frac{1}{2}\left[b - 1 - \sqrt{(b-1)^2 + 4a}\,\right].$$

Theorem 3. *For the mapping F_1 the following equality holds*

$$\dim_L K = 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)},$$

where

$$\alpha_1(x_-) = \sqrt{x_-^2 + b} - x_-.$$

Proof. The Jacobian matrix $T_x F_1$ of the mapping F_1 is as follows

$$\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}.$$

Let us show that the singular values of this matrix are the following

$$\begin{aligned} \alpha_1(x) &= \sqrt{x^2 + b} + |x|, \\ \alpha_2(x) &= \sqrt{x^2 + b} - |x| = \frac{b}{\alpha_1(x)}. \end{aligned} \tag{17}$$

Obviously

$$\begin{aligned} \alpha_1(x)^2 &= 2x^2 + b + 2|x|\sqrt{x^2 + b}\,, \\ \alpha_2(x)^2 &= 2x^2 + b - 2|x|\sqrt{x^2 + b}\,. \end{aligned}$$

Hence we can see that $\alpha_k(x)^2$ are the zeros of the polynomial

$$\lambda^2 - (4x^2 + 2b)\lambda + b^2,$$

which is a characteristic polynomial of the matrix

$$\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}.$$

Thus, formulas (17) are proved.

¿From Theorem 1 it follows that if there exists a number $s \in [0, 1)$ and a continuously differentiable positive on K function $p(\xi)$,

$$\xi = \begin{pmatrix} x \\ y \end{pmatrix},$$

such that the following inequality holds

$$\sup_{\xi \in K} \alpha_1(x)\alpha_2(x)^s \frac{p(F_1(\xi))}{p(\xi)} < 1, \tag{18}$$

then we have

$$\dim_L K \leq 1 + s.$$

Put

$$p(\xi) = e^{\gamma(1-s)(x+\sqrt{b}\ y)},$$

where γ is a positive running parameter.

We can see that the relation

$$\frac{p(F_1(\xi))}{p(\xi)} = e^{\gamma(1-s)(a+(b-1)x-x^2)}.$$

is satisfied. Whence it follows that taking the logarithm, condition (18) takes the form

$$\begin{aligned}&\sup_K\,[\ln\alpha_1(x) + s\ln\alpha_2(x) + \gamma(1-s)(a+(b-1)x-x^2)] = \\ &= \sup_K\,[(1-s)\ln\alpha_1(x) + s\ln b + \gamma(1-s)(a+(b-1)x-x^2)] < 0.\end{aligned}$$

This inequality is valid if

$$s\ln b + (1-s)\varphi(x) < 0, \quad \forall\, x \in (-\infty, +\infty),$$

where

$$\varphi(x) = \ln\left[\sqrt{x^2+b} + |x|\right] + \gamma(a+(b-1)x-x^2).$$

The inequalities $\gamma > 0$, $b-1 < 0$ imply the following estimate

$$\varphi(-|x|) \geq \varphi(|x|).$$

Therefore it is sufficient to consider the extremal points of the function $\varphi(x)$ for $x \in (-\infty, 0]$. In this set we have

$$\varphi'(x) = \frac{-1}{\sqrt{x^2+b}} + \gamma[(b-1)-2x], \quad \varphi''(x) < 0.$$

Choosing

$$\gamma = \frac{1}{(b-1-2x_-)\sqrt{x_-^2+b}},$$

we obtain: $\varphi'(x_-) = 0$. Consequently, under such a choice of γ the inequality

$$\varphi(x) \leq \left(\sqrt{x_-^2+b} + |x_-|\right) = \ln\alpha_1(x_-)$$

is satisfied. Thus, inequality (18) is true for all s, which satisfy the condition

$$s > \frac{\ln \alpha_1(x_-)}{\ln \alpha_1(x_-) - \ln b}. \tag{19}$$

Consequently, the following estimate

$$\dim_L K \le 1 + s$$

is valid for all s, satisfying inequality (19). Passing to the limit, we obtain the following inequality

$$\dim_L K \le 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}. \tag{20}$$

Note that the point $x = x_-$, $y = \sqrt{b}\, x_-$ is a fixed point of the mapping F_1. In this case we get

$$\alpha_1(x_-)\alpha_2(x_-)^s = 1,$$

$$s = \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}.$$

This implies that a local Lyapunov dimension of this stationary point is equal to

$$1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}. \tag{21}$$

Relations (20) and (21) result in the statement of Theorem 3.

Let us discuss the result obtained.

The running Lyapunov factor $p(\xi)$ makes it possible to compute the Lyapunov dimension of any invariant set K, containing the point $x = x_-$, $y = \sqrt{b}\, x_-$. Without this factor (more precisely, for $p(\xi) \equiv 1$) it was necessary to use the localization of invariant sets in order to obtain the upper estimates of the Lyapunov dimension [12, 22] only.

In many cases for the Lorenz system a situation turns out to be the same. The use of the Lyapunov functions makes it possible to perform the precise calculation of the Lyapunov dimension of invariant sets for this system too.

Consider the Lorenz system

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{22}$$

where r, b, σ are positive numbers.

Suppose, the following inequalities hold: $r > 1$,

$$b \ge 2, \quad \sigma + 1 \ge b. \tag{23}$$

Note that the strange attractors of system (22) were found by means of computer experiments for parameters $b = 8/3$, $\sigma = 10$, $r = 28$ and $b = 4$, $\sigma = 16$, $r = 40$ [3–6].

Under the assumption (23) by using the linear change of variables

$$x \to ax, \quad y \to \frac{a(b-1)}{\sigma}\,x + y, \quad z \to z,$$

where

$$a = \frac{\sigma}{\sqrt{2\sigma + (b-1)(\sigma - b)}}\,,$$

system (22) takes the form

$$\begin{aligned}
\dot{x} &= (b-\sigma-1)x + \frac{\sigma}{a}\,y,\\
\dot{y} &= \frac{\sigma}{a}\,x - by - axz, \\
\dot{z} &= -bz + \frac{a^2(b-1)}{\sigma}\,x^2 + axy.
\end{aligned} \tag{24}$$

Consider a shift operator along the trajectories of system (24) G_T, where T is an arbitrary positive number. Let K be an invariant set with respect to this operator G_T. Assume also that a set K contains the stationary point $x = y = z = 0$.

We now find a formula for the Lyapunov dimension $\dim_L K$ of set K with respect to a sequence of mappings G_T^i.

Theorem 4. *Let inequalities (23) and*

$$r\sigma^2(4-b) + 2\sigma(b-1)(2\sigma - 3b) > b(b-1)^2 \tag{25}$$

be valid. Then

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}\,. \tag{26}$$

Proof. The Jacobian matrix of the right side of system (24) is as follows

$$J = \begin{pmatrix} (b-\sigma-1) & \sigma/a & 0 \\ (\sigma/a) - az & -b & -ax \\ \dfrac{2a^2(b-1)}{\sigma}\,x + ay & ax & -b \end{pmatrix}.$$

For the following matrix

$$\frac{1}{2}\,(J^* + J) = \begin{pmatrix} (b-\sigma-1) & \dfrac{\sigma}{a} - \dfrac{a}{2}\,z & \dfrac{a^2(b-1)}{\sigma}\,x + ay \\ \dfrac{\sigma}{a} - \dfrac{a}{2}\,z & -b & 0 \\ \dfrac{a^2(b-1)}{\sigma}\,x + \dfrac{a}{2}\,y & 0 & -b \end{pmatrix}$$

we write its characteristic polynomial:

$$(\lambda+b)\Big\{\Big[\lambda^2+(\sigma+1)\lambda+b(\sigma+1-b)-\Big(\frac{\sigma}{a}-\frac{az}{2}\Big)^2\Big]-$$

$$-\Big[\frac{a^2(b-1)}{\sigma}\,x+\frac{ay}{2}\Big]^2\Big\}.$$

Whence it follows that the eigenvalues of the matrix $(J^*+J)/2$ take the form

$$\lambda_2(x,y,z)=-b,$$

$$\lambda_{1,3}(x,y,z)=-\frac{\sigma+1}{2}\pm\frac{1}{2}\Big((\sigma+1-2b)^2+$$

$$+\Big(\frac{2\sigma}{a}-az\Big)^2+\Big(\frac{2a^2(b-1)}{\sigma}\,x+ay\Big)^2\Big)^{1/2}.$$

By (23) $\lambda_1(x,y,z)\ge\lambda_2(x,y,z)\ge\lambda_3(x,y,z)$.

Consider a function of the Lyapunov type

$$v(x,y,z)=\frac{1}{2}\,a\theta^2(1-s)\left(\gamma_1x^2+\gamma_2\left((y^2+z^2+\frac{2a(b-1)}{\sigma}\,xy\right)+\gamma_3z\right),$$

where $s\in(0,1)$,

$$\theta^2=\left(2\sqrt{(\sigma+1-2b)^2+\Big(\frac{2\sigma}{a}\Big)^2}\right)^{-1},$$

$$\gamma_3=-\frac{4\sigma}{ab},\quad \gamma_2=\frac{a}{2},$$

$$\gamma_1=-\frac{1}{2\sigma}\left[2\gamma_2a^2\,\frac{(r\sigma-b^2+1)}{\sigma}+\gamma_3a^2+4\,\frac{a^3(b-1)}{\sigma}\right].$$

Consider also an expression

$$\begin{aligned}&2[\lambda_1(x,y,z)+\lambda_2(x,y,z)+s\lambda_3(x,y,z)+\dot v(x,y,z)]=\\&=-(\sigma+1+2b)-s(\sigma+1)+(1-s)\varphi(x,y,z),\end{aligned}$$

where

$$\varphi(x,y,z) = \left((\sigma+1-2b)^2 + \left(\frac{2\sigma}{a} - az\right)^2 + \left(\frac{2a^2(b-1)}{\sigma}x + ay\right)^2\right)^{1/2} +$$

$$+\theta^2\Big\{ \left(2a\gamma_1(b-\sigma-1) + 2\gamma_2 a(b-1) + \gamma_3 a^3\frac{b-1}{\sigma}\right)x^2 - 2a\gamma_2 y^2 -$$

$$-2a\gamma_2 bz^2 + \left(2\sigma\gamma_1 + 2\gamma_2\left(a^2\frac{r\sigma - b^2+1}{\sigma}\right) + \gamma_3 a^2\right)xy - \gamma_3 abz\Big\}.$$

Using the obvious inequality

$$\sqrt{u} \le \frac{1}{4\theta^2} + \theta^2 u,$$

we obtain an estimate

$$\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2\Big\{(\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2 +$$

$$+\left[2a\gamma_1(b-\sigma-1) + 2\gamma_2 a(b-1) + \gamma_3 a^3\frac{b-1}{\sigma} + \frac{4a^4(b-1)^2}{\sigma^2}\right]x^2 +$$

$$+[a^2 - 2a\gamma_2]y^2 + [a^2 - 2\gamma_2 ab]z^2 + \Big[2\gamma_1\sigma + 2\gamma_2 a^2\frac{r\sigma - b^2+1}{\sigma} +$$

$$+\gamma_3 a^2 + 4a^3\frac{b-1}{\sigma}\Big\}xy - [\gamma_3 ab + 4\sigma]z\Big\}.$$

Note that parameters γ_1, γ_2 and γ_3 are chosen in such a way that they satisfy the relation

$$\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2\Big\{(\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2 +$$

$$+\Big[2a\gamma_1(b-\sigma-1) + 2\gamma_2 a(b-1) + \gamma_3\frac{a^3(b-1)}{\sigma} +$$

$$+\frac{4a^4(b-1)^2}{\sigma^2}\Big]x^2\Big\}.$$

It is not difficult to prove that for the above parameters $\gamma_1, \gamma_2, \gamma_3$ conditions (25) result in the equality

$$2a\gamma_1(b-\sigma-1) + 2\gamma_2 a(b-1) + \gamma_3\frac{a^3(b-1)}{\sigma} + \frac{4a^4(b-1)^2}{\sigma^2} \le 0.$$

Thus, for all x, y, z

$$\varphi(x,y,z) \le \sqrt{4r\sigma + (\sigma-1)^2}\ .$$

Whence it follows that for any number

$$s < s_0 = \frac{\sqrt{4r\sigma + (\sigma-1)^2} - 2b - \sigma - 1}{\sqrt{4r\sigma + (\sigma-1)^2} + \sigma + 1}$$

there exists a number $\varepsilon > 0$ such that for all x, y, z the following estimate holds

$$\lambda_1(x,y,z) + \lambda_2(x,y,z) + s\lambda_3(x,y,z) + \dot{v}(x,y,z) < -\varepsilon.$$

Letting s tend to s_0 from the right and using Theorem 2, we obtain an estimate

$$\dim_L K \leq 3 - \frac{2(\sigma+b+1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{27}$$

It is easy to see that the local Lyapunov dimension of the stationary point $x = y = z = 0$ of system (24) is equal to

$$3 - \frac{2(\sigma+b+1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{28}$$

Relations (27) and (28) imply formula (26).

We make the following remark concerning the statements of Theorems 3 and 4.

It is clear that the quantity $\alpha_1(x_-)$ is also an eigenvalue of the Jacobian matrix for the Hennon mapping at the point $x = x_-$, $y = y_-$.

Using formula (26), we can see that $\sigma + b + 1$ is a spur of the Jacobian matrix for the right side of system (22) and the number

$$\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}$$

is an eigenvalue of the Jacobian matrix for the right side of system (22) at the point $x = y = z = 0$.

Thus, the quantities

$$\beta_1 = 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)}$$

and

$$\beta_2 = 3 - \frac{2(\sigma+b+1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}$$

are invariants with respect to the linear nondegenerate transformations in the phase spaces of the Hennon and Lorenz systems. Therefore the estimates

$$\dim_H K \leq \dim_F K \leq \beta_j$$

are invariant with respect to the linear nonsingular transformations.

Consider the Rössler system

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x, \\ \dot{z} &= -bz + a(y - y^2), \end{aligned} \tag{29}$$

where a and b are positive numbers.

Theorem 5. *The invariant set K of system (29) satisfies the estimate*

$$\dim_L K \le 3 - \frac{2b}{b + \sqrt{(a+2b)^2 + b^2 + 1}}. \tag{30}$$

Proof. The Jacobian matrix for the right side of system (29) have the form

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & a(1-2y) & -b \end{pmatrix}.$$

Therefore the characteristic polynomial of the symmetrized matrix

$$\frac{1}{2}(J + J^*) = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & a(1-2y)/2 \\ -1/2 & a(1-2y)/2 & -b \end{pmatrix}$$

is as follows

$$\lambda^3 + b\lambda^2 - \frac{1 + a^2(1-2y)}{4}\lambda = 0.$$

Hence the eigenvalues $\lambda_j(x, y, z)$ of the matrix $(J^* + J)/2$ are the following

$$\lambda_1 = \frac{1}{2}\left[-b + \sqrt{b^2 + 1 + a^2(1-2y)^2}\right],$$

$$\lambda_2 = 0,$$

$$\lambda_3 = \frac{1}{2}\left[-b - \sqrt{b^2 + 1 + a^2(1-2y)^2}\right].$$

We can easily see that $\lambda_1 > \lambda_2 > \lambda_3$ for all y.

Consider an expression

$$\begin{aligned} \lambda_1 + \lambda_2 + s\lambda_3 + \dot{v} = \frac{1}{2}[-(1+s)b+ \\ +(1-s)(b^2 + 1 + a^2(1-2y)^2)^{1/2} + 2\dot{v}], \end{aligned} \tag{31}$$

where $s \in [0, 1)$.

Now we choose the function v of the Lyapunov type in the form

$$v = \frac{1}{2}(1 - s)æ(z - bx).$$

Here æ is a running parameter. In this case expression (31) takes the form

$$\lambda_1 + \lambda_2 + s\lambda_3 + \dot{v} = \frac{1}{2}\,[-(1+s)b + (1-s)\varphi(y, æ)],$$

where

$$\varphi(y, æ) = \sqrt{b^2 + 1 + a^2(1-2y)^2} + æ[(a+b)y - ay^2].$$

For any number $\theta \neq 0$ the following relations hold

$$\varphi(y, æ) \leq \theta^2[b^2 + 1 + a^2(1-2y)^2] + \frac{1}{4\theta^2} +$$

$$+æ[(a+b)y - ay^2] = \theta^2(a^2 + b^2 + 1) + \frac{1}{4\theta^2} -$$

$$-(æa - 4\theta^2 a^2)y^2 - [4\theta^2 a^2 - æ(a+b)]y.$$

If we choose variable parameters so that $æa > 4\theta^2 a^2$, we obtain an inequality

$$\varphi(y, æ) \leq \theta^2(a^2 + b^2 + 1) + \frac{1}{4\theta^2} + \frac{[4\theta^2 a^2 - æ(a+b)]^2}{4(æa - 4\theta^2 a^2)}.$$

This implies that for

$$æ = 4\theta^2 a\,\frac{a+2b}{a+b}, \quad \theta^2 = \frac{1}{2\sqrt{(a+2b)^2 + b^2 + a}}.$$

the following estimate holds

$$\varphi(y, æ) \leq \sqrt{(a+b)^2 + b^2 + 1}\ .$$

Thus, for any positive s, satisfying inequality

$$s < s_0 = 1 - \frac{2b}{b + \sqrt{(a+2b)^2 + b^2 + 1}},$$

there exists a number $\varepsilon > 0$ such that it satisfies the inequality

$$\lambda_1 + \lambda_2 + s\lambda_3 + \dot{v} \leq -\varepsilon$$

for all x, y, z.

By Theorem 2 and passing to the limit as $s \to s_0$, we obtain estimate (30).

Note that without using the Lyapunov functions (i.e., assuming that $v \equiv$ const) for system (29) it is impossible to obtain nontrivial estimates of the Lyapunov dimension starting from the condition

$$\lambda_1 + \lambda_2 + s\lambda_2 < 0.$$

For $s < 1$ and as $y \to \infty$ we have

$$\lambda_1 + \lambda_2 + s\lambda_3 \to +\infty.$$

Note also that unlike the Hennon and Lorenz systems we obtain here only the upper estimate of the Lyapunov dimension. The precise computation of Lyapunov dimension for the Rössler systems is yet an unsolved problem.

In addition, note that now for estimating the Lyapunov dimension the frequency method is developed, which is based on the use of the Lyapunov functions of the "quadratic form" type. Such an approach is described in the work [23].

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