THE BROCKETT STABILIZATION PROBLEM

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In the work [1] R. Brockett has formulated the following problem.

Given a triple constant matrices A, B, C under what circumstances does there exist a time-dependent matrix $K(t)$ such that the system

$$\frac{dx}{dt} = Ax + BK(t)Cx, \quad x \in \mathbb{R}^n \tag{1}$$

is asymptotically stable.

In [1] it was noted that the problem is still interesting and unsolved in the case when $K(t)$ is a scalar function.

Notice that a stabilization of system (1) with help of a constant matrix K is a classical problem of the control theory. A reformulation of the Brockett problem from this point of view is in following.

How much would an application of a time dependent matrix $K(t)$ extend the capabilities of stabilization?

In a number of cases the answer to this question can be found by means of the following algorithm for stabilization.

Suppose that there exists a matrix K_1 such that a system

$$\frac{dx}{dt} = (A + B\mu K_1 C)x \tag{2}$$

with a scalar parameter μ possesses a stable linear invariant manifold $L(\mu)$ for $\mu \geq \mu_0$. Here μ_0 is some number.

We also assume that

$$\lim_{\mu\to+\infty} L(\mu) = L_0 \tag{3}$$

and for any number $\delta > 0$ there exists $\mu \geq \mu_0$ such that

$$|x(1, x_0)| \leq \delta, \quad \forall\, x_0 \in \{|x_0| = 1\} \cap L(\mu). \tag{4}$$

Here $x(0, x_0) = x_0$ and a limit (3) is understood in the sense that $L(\mu) \cap \{|x| \leq 1\}$ is situated in a ε-neighborhood of $L_0 \cap \{|x| \leq 1\}$, where $\varepsilon \to 0$ as $\mu \to +\infty$.

The assumption we have formulated means a fast convergence of trajectories on manifold $L(\mu)$ for large parameter μ.

Let us denote by $M(\mu)$ a linear invariant manifold of system (2), for which we have

$$\lim_{\mu\to+\infty} M(\mu) = M_0,$$

$$\dim M(\mu) + \dim L(\mu) = n, \quad M(\mu) \cap L(\mu) = \{0\}.$$

Suppose that $M(\mu)$ is a manifold of slow movements, i.e., there exists a number R_1 such that for all $\mu \geq \mu_0$ the inequality holds

$$|x(1,x_0)| \leq R_1, \quad \forall\, x_0 \in \{|x_0| = 1\} \cap M(\mu). \tag{5}$$

Now assume that there exists a matrix K_2 such that for the system

$$\frac{dy}{dt} = (A + BK_2C)y \tag{6}$$

a number τ exists such that

$$Y(\tau)M_0 \subset L_0. \tag{7}$$

Here $Y(t)$ is a fundamental matrix of system (6), $Y(0) = I$.

Let us denote a $(2+\tau)$-periodic matrix $K(t)$ in the following way

$$K(t) = \begin{cases} \mu K_1 & \text{for} \quad t \in [0,1), \\ K_2 & \text{for} \quad t \in [1, 1+\tau), \\ \mu K_1 & \text{for} \quad t \in [1+\tau, 2+\tau), \end{cases} \tag{8}$$

Theorem 1. *A system* (1) *with a matrix* $K(t)$ *of the type* (8) *for sufficiently large* μ *is asymptotically stable.*

Proof. The construction of manifolds L_0 and M_0 implies that for any number $\varepsilon > 0$ there exists $\mu_1 \geq \mu_0$ satisfying the following property.

For $t = 1$ the solution $x(t,x_0)$ of system (1) with $\mu \geq \mu_1$ and with the initial data $x_0 = x(0,x_0)$, $|x_0| = 1$ is situated in a ε-neighborhood of manifold

$$M_0 \cap \{|x| \leq R_1\}.$$

It is clear that there exists a number R_2 having the following property. For $t = 1+\tau$ the solution $x(t,x_0)$ with the initial data $x_0 = x(0,x_0)$, $|x_0| = 1$ is situated in an $R_2\varepsilon$-neighborhood of manifold

$$L_0 \cap \{|x| \leq R_2\}.$$

It can readily be seen that for $t = 2+\tau$ for the solution considered the following estimate

$$|x(2+\tau, x_0)| \leq \nu\varepsilon$$

is valid if μ is sufficiently large number, ν is some number.

Choosing ε sufficiently small (and therefore μ sufficiently large) we obtain that for all x_0 from a sphere $|x_0| = 1$ the following estimate

$$|x(2+\tau, x_0)| < \frac{1}{2}$$

is valid. The latter means an asymptotical stability of system (1) with a periodical matrix $K(t)$ of the type (8).

Theorem 1 implies some corollaries.

Let us consider a periodic solution $z(t)$ of system

$$\dot{z} = Qz, \quad z \in \mathbb{R}^n$$

where Q is an $(n \times n)$-matrix such that $\det Q \neq 0$.

Lemma. *For any vector $h \in \mathbb{R}^n$ there exists a number τ such that $h^* z(\tau) = 0$.*

Proof. Let $h^* z(t) > 0 \quad \forall t \in \mathbb{R}^n$. Then the periodicity of function $h^* z(t)$ implies the following relation

$$\lim_{t \to +\infty} \int_0^t h^* z(t)\, dt = +\infty.$$

On the other hand, we have

$$\int_0^t h^* z(t)\, dt = h^* Q^{-1}(z(t) - z(0)).$$

Whence it follows that the integral is a bounded function for $t \geq 0$. This contradiction proves the existence of a number τ such that $h^* z(\tau) = 0$.

Theorem 2. *Let there exist matrices K_1 and K_2 satisfying the following conditions:*
1) *a matrix BK_1C has $n-1$ eigenvalues with negative real parts and* $\det BK_1C = 0$,
2) *for a vector $u \neq 0$, satisfying an equality $BK_1Cu = 0$, and for a certain number λ, the vector-function*

$$\exp\left[(A + BK_2C + \lambda I)t\right]u \tag{9}$$

is periodic.

Then there exists a periodic matrix $K(t)$ such that system (1) *is asymptotically stable.*

To prove this theorem it is necessary to check a satisfaction of properties (3)–(7).

¿From hypothesis 1) of Theorem 2 the properties (3)–(5) follow. Here hyperplane L_0 is an $(n-1)$-dimensional stable manifold of the following linear system

$$\frac{dy}{dt} = BK_1Cy.$$

¿From hypothesis 2) of Theorem 2 and the lemma it follows that there exists τ such that

$$\exp\left[(A + BK_2C)\tau\right]u \in L_0. \tag{10}$$

Thus, we can see that by Theorem 1 system (1) with a matrix of the type (8) is asymptotically stable.

Theorem 3. *Let $n = 2$ and there exist matrices K_1, K_2, and number λ such that the following conditions hold:*
1) $\det BK_1C = 0$ *and* $\operatorname{Tr} BK_1C \neq 0$,
2) *a matrix $A + BK_2C + \lambda I$ has purely imaginary eigenvalues.*

Then there exists a periodic matrix $K(t)$ such that system (1) *is asymptotically stable.*

In order to prove this it should be noted that hypothesis 1) of Theorem 3 implies the existence of a nonzero eigenvalue of matrix BK_1C. If this eigenvalue is negative, then the hypothesis 1) of Theorem 2 is satisfied. If it is positive, then the hypothesis 1) of Theorem 2 is satisfied for the matrix $-K_1$.

¿From the hypothesis 2) of Theorem 3 it follows that the hypothesis 2) of Theorem 2 is satisfied. Theorem 3 is proved.

We now consider a case that $n = 2$, B is a column vector and C is a row vector. In this case the transfer function $W(p)$ of system (1) takes the form

$$W(p) = C(A - pI)^{-1}B = \frac{\rho p + \gamma}{p^2 + \alpha p + \beta}.$$

Let $\rho \neq 0$. Then, we can assume without loss of generality that $\rho = 1$. Let us also suppose that the inequality holds

$$\gamma^2 - \alpha\gamma + \beta \neq 0.$$

This condition assures a nondegeneracy of transfer function $W(p)$. It is well known [2] that in this case system (1) can be reduced to the following form

$$\begin{aligned} \dot{\sigma} &= \eta, \\ \dot{\eta} &= -\alpha\eta - \beta\sigma - K(t)(\eta + \gamma\sigma). \end{aligned} \tag{11}$$

Here $K(t)$ is a scalar function.

The problem on a stabilization with help of a constant value $K(t) \equiv K_0$ has a positive solution if and only if

$$\alpha + K_0 > 0, \quad \beta + \gamma K_0 > 0.$$

For the existence of a number K_0, satisfying these inequalities, it is necessary and sufficient that the inequalities either $\gamma > 0$ or $\gamma \leq 0$, $\alpha\gamma < \beta$ where satisfied.

Now we consider the case that a stabilization with help of a constant K_0 is impossible:

$$\gamma \leq 0, \quad \alpha\gamma > \beta.$$

Let us check the satisfaction of the hypotheses of Theorem 3. It can easily be seen that hypothesis 1) of Theorem 3 is satisfied. For a satisfaction of the hypothesis 2) of Theorem 3 it is necessary and sufficient that for some numbers k_2 and λ a polynomial

$$p^2 + (\alpha + k_2 - 2\lambda)p + (\beta + k_2\gamma - \lambda^2)$$

has a purely imaginary zeroes. This is valid in the case that

$$\alpha + k_2 - 2\lambda = 0, \quad \beta + k_2\gamma - \lambda^2 > 0. \tag{12}$$

For the existence of λ and k_2 both, satisfying condition (12), it is necessary and sufficient that the following inequality holds

$$\gamma^2 - \alpha\gamma + \beta > 0. \tag{13}$$

Thus, if inequality (13) is valid, then by Theorem 3 there exists a periodic function $K(t)$ such that system (11) is asymptotically stable.

This implies that if an inequality

$$\gamma^2 - \alpha\gamma + \beta < 0 \tag{14}$$

is valid, then the estimate

$$(\eta + \gamma\sigma)^\bullet = (-\gamma^2 - \beta + \alpha\gamma)\sigma > 0,$$

$\forall\,\sigma > 0$, $\eta = -\gamma\sigma$ is true. Whence it follows that for $\sigma(0) > 0$, $\eta(0) + \gamma\sigma(0) > 0$ the inequality holds

$$\eta(t) + \gamma\sigma(t) > 0, \quad \forall\, t \geq 0.$$

But then $\dot\sigma(t) = \eta(t) > 0$, $\forall\, t \geq 0$. Hence the instability of system (11) follows.

Thus, we have proved the following result.

Theorem 4. [3] *If inequality* (13) *holds, then system* (11) *is stabilized with help of a periodic function* $K(t)$*.*

If inequality (14) *holds, then there do not exist functions* $K(t)$*, for which system* (11) *is asymptotically stable.*

This theorem was proofed also by another class of functions $K(t)$:

$$K(t) = (K_0 + K_1\omega\cos\omega t), \quad \omega \gg 1$$

and averaging methods [4].

Let us consider system (1) with $n = 3$, C is row vector, B is column vector, $K(t)$ is function: $\mathbb{R}^1 \to \mathbb{R}^1$.

Theorem 5. *Let* $CB \neq 0$ *and there exist a numbers* K_1, K_2 *satisfying the following conditions:*

1) *a matrix* $A + K_1BC$ *has two complex eigenvalues and one negative eigenvalue,*

2) *function*

$$C\exp[(A + K_2BC)t]B$$

has at least one zero on interval $(-\infty, 0)$*.*

Then there exists a periodic function $K(t)$ *such that system* (1) *is asymptotically stable.*

References

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[4] Morean L., Aeyels D. Stabilization by means of periodic output feedback. Proceedings of Conference of Decision and Control. Phoenix, Arizona USA, Desember 1999. P.108–109.