Sufficient conditions for elliptic problem of optimal control in the Sobolev space $W_0^{1,2}(I)$, where I is a bounded interval in $\mathbb{R}$

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Abstract

This paper is concerned with the local minimization problem for a variety of non Frechet-differentiable Gâteaux functional $J(f) \equiv \int\limits_I v(x, u(x), f(x))dx$ in the Sobolev space $(W_0^{1,2}(I), \|.\|_p)$, where u is the solution of the Dirichlet problem for a linear uniformly elliptic operator with nonhomogenous term f and $\|.\|_p$ is the norm generated by the metric space $L^p(I)$, $(p > 1)$.We use a recent extension of Frechet-Differentiability (approach of Taylor mappings see [1]), and we give various assumptions on v to guarantee a critical point is a strict local minimum.
Finally, we give an example of a control problem, where classical Frechet differentiability can't be used and their approach of Taylor mappings works.

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1 Preliminaries

1.1 Description of the optimization problem

Let A be an elliptic operator of second order:
$Au \equiv \sum\limits_{|l|\leq 1,|s|\leq 1} (-1)^l \mathcal{D}^l(a_{ls}(x)\mathcal{D}^s u)$, where $a_{ls}(x) \in \mathcal{D}(\overline{I})$.
Suppose that I is a bounded interval in $\mathbb{R}$.
Let us consider the problem :

$$\begin{cases} Au = f, & (1.1) \\ u/_{\partial I} = 0. & (1.2) \end{cases}$$

For this problem, let us state Agmon's-Douglis-Niremberg's theorem:

Theorem 1.1 *Let $1 < q < \infty$, then we have*
$\forall f \in L^q(I)$, there exists a unique solution $u \in W^{2,q}(I) \bigcap W_0^{1,q}(I)$ of problem (1.1),(1.2). Moreover, $\forall m \geq 0$ if $f \in W^{m,q}(I)$, then $u \in W^{m+2,q}(I)$ and $\|u\|_{W^{m+2,q}(I)} \leq c\|f\|_{W^{m,q}(I)}$.

Let $f \in F \subset W_0^{1,2}(I)$ a control and let u the solution of problem (1.1),(1.2) in $W_0^{1,2}(I) \bigcap W^{2,2}(I)$ associated to f.
Let us consider $J_k(f) = \int\limits_I v_k(x, u(x), f(x))dx + c_k\|f\|^2_{W^{1,2}(I)}$, $(k = 0, 1, 2, ..., s_1)$
and $J_k(f) = \int\limits_I v_k(x, u(x), f(x))dx$, $(k = s_1 + 1, s_1 + 2, ..., s_1 + s_2)$,
where each function $v_k : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable on $I \times \mathbb{R} \times \mathbb{R}$ and has second derivative with respect to (u, f) on $\mathbb{R} \times \mathbb{R}$ for almost all $x \in I$.
Put $J = (J_{s_1+1}, \ldots\ldots., J_{s_1+s_2})$.
We consider three problems of minimizing the functional $J_0(f)$:

$$i) J_0(f) \;\rightarrow\; \min, \tag{1.3}$$

$$ii) J_0(f) \;\rightarrow\; \min,\; J(f) = 0, \tag{1.4}$$

$$iii) J_0(f) \;\rightarrow\; \min,\; J(f) = 0,\; J_k(f) \leq 0 \;\;, (k = 1, 2, ..., s_1). \tag{1.5}$$

We must choose a control f^0 in order that the solution u^0 of the problem (1.1),(1.2) with $f = f^0$ satisfies the inequality type : $J_k(f) \leq 0, (1 \leq k \leq s_1)$ and the equality type : $J_k(f) = 0, (s_1 + 1 \leq k \leq s_1 + s_2)$ and the functional $J_0(f)$ takes a minimum valued. This control f^0 will be called optimal.

1.2 Taylor mapping and lower semi-Taylor mapping

Let $\|.\|_{W^{1,2}(I)}$ the usual norm in $W_0^{1,2}(I)$, F a subset of $W_0^{1,2}(I)$, τ a topology in F, Y a normed space, and $\|.\|_Y$ a norm in Y.
According to [1], a mapping $r : F \longrightarrow Y$ (respectively, $r : F \longrightarrow \mathbb{R}$) is said to be infinitesimally $(\tau, \|.\|_{W^{1,2}(I)})$-small (respectively, infinitesimally lower $(\tau, \|.\|_{W^{1,2}(I)})$-semismall) of order p_1 at $f \in F$ if : $\forall\ \varepsilon > 0,\ \exists\ O_f \in \tau, \forall\ h\ \in W_0^{1,2}(I)$ we have

$$f + h \in O_f \Rightarrow \|r(h)\|_Y \leq \varepsilon \|h\|^{p_1}_{W^{1,2}(I)},$$

(respectively, $\forall\ \varepsilon > 0,\ \exists\ O_f \in \tau, \forall\ h\ \in W_0^{1,2}(I)$ we have

$$f + h \in O_f \Rightarrow r(h) \geq -\varepsilon \|h\|^{p_1}_{W^{1,2}(I)});$$

here and below, O_f is a neighborhood of f in (F, τ).
A mapping $J : F \to Y$ (respectively, $J : F \to \mathbb{R}$) is called a $(\tau, \|.\|_{W^{1,2}(I)})$-Taylor (respectively, lower $(\tau, \|.\|_{W^{1,2}(I)})$-semi-Taylor) mappings of order p_1 at $f \in F$ if there exist k -linear symmetric (not necessarily continuous) mappings $J^{(k)}(f) : (W_0^{1,2}(I))^k \to Y$(respectively, $J^{(k)}(f) : (W_0^{1,2}(I))^k \to \mathbb{R},\ k = 1, .., p_1$, such that

$$J(f + h) - J(f) = J^{(1)}(f)h + 2^{-1}J^{(2)}(f)(h, h) + ...(p_1)!^{-1}J^{(p_1)}(f)(h, ...h) + r(h),$$

where $r : F \longrightarrow Y$ (respectively, $r : F \longrightarrow \mathbb{R}$) is an infinitesimally $(\tau, \|.\|_{W^{1,2}(I)})$ -small (respectively, infinitesimally lower $(\tau, \|.\|_{W^{1,2}(I)})$ -semismall) mappings of order p_1 at $f \in F$.
We note that $J^{(1)}(f), ..., J^{(p_1)}(f)$ are not in general single-valued.
The set of tuples $(J^{(1)}(f), ..., J^{(p_1)}(f))$ is denoted by $S_n(J, f)$.
Let us solve the problems (1.3),(1.4) and (1.5).
For the problem (1.5) let us introduce the Lagrange functions:

$$\mathcal{L}(f, y^*, \lambda, \lambda_0) = \sum_{k=0}^{s_1} \lambda_k J_k(f) + < y^*, J(f) >, \quad (1.6)$$

$$\mathcal{L}_f(f, y^*, \lambda, \lambda_0) = \sum_{k=0}^{s_1} \lambda_k J_k^{(1)}(f) + < y^*, J^{(1)}(f) >, \quad (1.7)$$

$$\mathcal{L}_{ff}(f, y^*, \lambda, \lambda_0) = \sum_{k=0}^{s_1} \lambda_k J_k^{(2)}(f) + < y^*, J^{(2)}(f) >, \quad (1.8)$$

where $\lambda_0 \in \mathbb{R},\ y^* \in (\mathbb{R}^{s_2})^*,\ \lambda \in (\mathbb{R}^{s_1})^*$.

Also for the problem (1.4), let us introduce the Lagrange functions :

$\mathcal{L}(f, y^*, \lambda_0) = \lambda_0 J_0(f) + < y^*, J(f) >,$ (1.9)

$\mathcal{L}_f(f, y^*, \lambda_0) = \lambda_0 J_0^{(1)}(f) + < y^*, J^{(1)}(f) >,$ (1.10)

$\mathcal{L}_{ff}(f, y^*, \lambda_0) = \lambda_0 J_0^{(2)}(f) + < y^*, J^{(2)}(f) >,$ (1.11)

where $\lambda_0 \in \mathbb{R},\ y^* \in (\mathbb{R}^{s_2})^*$.

Let us give the following lemma where the proof can be traced back to[1].

Lemma 1.1 *Let $(\Omega, \sum, \mu)$ be a measure space with σ- finite measure, and let X be a complete linear metric space continuously imbedded in the metric space $M(\Omega)$ of equivalence classes of measurable almost everywhere finite functions $x : \Omega \longrightarrow \mathbb{R}$, with the metrizable topology $\tau(meas)$ of convergence in measure on each set of Σ finite measure.*

Suppose that X contains with each element $x(s)$ the function $\mid x(s) \mid$, the metric in X is translation-invariant, and $\rho(x,0) = \rho(\mid x \mid, 0)$ for each $x \in X$. Then for each sequence $x_n \to 0$ in X there exist a subsequence x_{n_k} and an element $z \in X$ such that : $\mid x_{n_k}(s) \mid \leq z(s)$, k=1,2,.... in the sense of the natural order on classes of functions.

2 Sufficient conditions of local minimum for Gâteaux functional of second order Dirichlet problem

Let I a bounded interval in $\mathbb{R}$ and let F a subset of $W_0^{1,2}(I)$.

Let G the functional defined on F by: $G(f) = \int\limits_I v(x, u(x), f(x))dx$, where $u(x)$ is the solution of problem (1.1),(1.2) in $W_0^{1,2}(I) \bigcap W^{2,2}(I)$ and the function $v : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable on $I \times \mathbb{R} \times \mathbb{R}$ and has second derivative with respect to (u, f) on $\mathbb{R} \times \mathbb{R}$ for almost all $x \in I$. Suppose also that $v, v_{uf}^{(2)},\ v_{fu}^{(2)}$ are continuous in $I \times \mathbb{R} \times \mathbb{R}$.

Let τ_p the topology generated by the metric space $L^p(I)$, where $p > 1$.

In the rest of this section $a = const.$

Theorem 2.1 *Suppose that with every requirements of paragraph (1) and (2) the following conditions are verified : $v_{uf}^{(2)},\ v_{fu}^{(2)}$ are continuous in $I \times \mathbb{R} \times \mathbb{R}$, $p > 1$.*

Let us suppose also that

$$\begin{aligned} \mid v(x,u,f)\mid + \mid v_u^{(1)}(x,u,f)\mid + \mid v_f^{(1)}(x,u,f)\mid &\leq a\Big(\mid u\mid^{p+2} + \mid f\mid^{p+2}\Big) + b_5(x) \\ \mid v_{uu}^{(2)}(x,u,f)\mid +2\mid v_{uf}^{(2)}(x,u,f)\mid + \mid v_{ff}^{(2)}(x,u,f)\mid &\leq a\Big(\mid u\mid^{p} + \mid f\mid^{p}\Big) + b_6(x) \end{aligned}$$

where $b_5 \in L_1(I)$, $b_6 \in L_1(I)$.
then G *is* $(\tau_p, \|.\|_{W_2^1(I)})$ *Taylor mapping of second order at every* $f \in F$.
Moreover $G^{(2)}(f) \in \mathcal{B}((W_0^{1,2}(I), \|.\|_{W_2^1(I)}), \mathbb{R})$
and $G^{(1)}(f) \in \mathcal{L}((W_0^{1,2}(I), \|.\|_{W_2^1(I)}), \mathbb{R})$

Proof

let us prove first that the functional G is finite .

We have

$$\begin{aligned} |G(f)| &= \left|\int_I v(x,u,f)dx\right| \\ &\leq \int_I |v(x,u,f)|dx \\ &\leq a\Big(\int_I |u(x)|^{p+2}\,dx + \int_I |f(x)|^{p+2}\,dx\Big) + \int_I b_5(x)dx \\ &\leq a\Big(\|u(x)\|_{L_{p+2}(I)}^{p+2} + \|f(x)\|_{L_{p+2}(I)}^{p+2}\Big) + \|b_5(x)\|_{L_1(I)} \\ &\leq c_1\Big(\|u(x)\|_{W_2^1(I)}^{p+2} + \|f(x)\|_{W_2^1(I)}^{p+2}\Big) + \|b_5(x)\|_{L_1(I)} \\ &< \infty. \end{aligned}$$

thus the functional G is finite.

$$\begin{aligned} \text{let} \quad R: \; W_0^{1,2}(I) &\longrightarrow W_0^{1,2}(I) \\ h &\longmapsto (R(h))(x) \end{aligned}$$

where $(R(h))(x)$ is a solution of problem

$$\begin{cases} Au = h & (2.1) \\ u/_{\partial I} = 0 & (2.2) \end{cases}$$

such a solution exists $\forall h \in W_0^{1,2}(I)$.

let $G^{(1)}(f)$ defined by :

$$G^{(1)}(f)h = \lim_{\lambda\to 0} \lambda^{-1}\Big(G(f+\lambda h) - G(f)\Big)$$

$$
\begin{aligned}
&= \lim_{\lambda\to 0} \lambda^{-1} \int\limits_I \Big[v(x,u+\lambda R(h), f+\lambda h) - v(x,u,f)\Big]dx \\
&= \lim_{\lambda\to 0} \lambda^{-1} \int\limits_I \Big[v(x,u+\lambda R(h), f+\lambda h) - v(x,u,f+\lambda h) + v(x,u,f+\lambda h) \\
&\quad -v(x,u,f)\Big]dx \\
&= \lim_{\lambda\to 0} \int\limits_I \Big[\int_0^1 v_u^{(1)}(x,u+\theta\lambda R(h), f+\lambda h)R(h)d\theta + \int_0^1 v_f^{(1)}(x,u,f+\rho\lambda h)hd\rho\Big]dx \\
&= \lim_{\lambda\to 0} \int\limits_I \Big[\int_0^1 \Big[v_u^{(1)}(x,u+\theta\lambda R(h), f+\lambda h) - v_u^{(1)}(x,u,f)\Big]R(h)d\theta \\
&\quad + \int_0^1 v_u^{(1)}(x,u,f)R(h)d\theta + \int_0^1 \Big[v_f^{(1)}(x,u,f+\rho\lambda h) - v_f^{(1)}(x,u,f)\Big]hd\rho \\
&\quad +h\int_0^1 v_f^{(1)}(x,u,f)d\rho\Big]dx \\
&= \int\limits_I v_u^{(1)}(x,u,f)R(h)dx + \int\limits_I v_f^{(1)}(x,u,f)hdx
\end{aligned}
$$

therefore: $$G^{(1)}(f) = \int\limits_I v_u^{(1)}(x,u,f)R(h)dx + \int\limits_I v_f^{(1)}(x,u,f)hdx. \qquad (2.3)$$

$$
\begin{aligned}
G^{(2)}(f)(h_1,h_2) &= \lim_{\lambda\to 0} \lambda^{-1}\Big[G^{(1)}(f+\lambda h_2) - G^{(1)}(f)\Big]h_1 \\
&= \lim_{\lambda\to 0} \lambda^{-1}\Bigg[\int\limits_I \Big[v_u^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_u^{(1)}(x,u,f)\Big]R(h_1)dx \\
&\quad + \int\limits_I \Big[v_f^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_f^{(1)}(x,u,f)\Big]h_1 dx\Bigg] \\
&= \lim_{\lambda\to 0} \lambda^{-1}\Bigg[\int\limits_I \Big[v_u^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_u^{(1)}(x,u,f+\lambda h_2) \\
&\quad +v_u^{(1)}(x,u,f+\lambda h_2) - v_u^{(1)}(x,u,f)\Big]R(h_1)dx \\
&\quad + \int\limits_I \Big[v_f^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_f^{(1)}(x,u,f+\lambda h_2) \\
&\quad +v_f^{(1)}(x,u,f+\lambda h_2) - v_f^{(1)}(x,u,f)\Big]h_1 dx\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\lambda\to 0}\lambda^{-1}\Big[\int\limits_I \Big[\int_0^1 v^{(2)}_{uu}(x,u+\theta\lambda R(h_2), f+\lambda h_2)\lambda R(h_2)d\theta \\
&\quad + \int_0^1 v^{(2)}_{fu}(x,u,f+\rho\lambda h_2)\lambda h_2 d\rho\Big] R(h_1)dx \\
&\quad + \int\limits_I \Big[\int_0^1 v^{(2)}_{uf}(x,u+\theta\lambda R(h_2), f+\lambda h_2)\lambda R(h_2)d\theta \\
&\quad + \int_0^1 v^{(2)}_{ff}(x,u,f+\rho\lambda h_2)\lambda h_2 d\rho\Big] h_1 dx\Big] \\
&= \int\limits_I v^{(2)}_{uu}(x,u,f)R(h_1)R(h_2)dx + \int\limits_I v^{(2)}_{uf}(x,u,f)R(h_1)h_2dx \\
&\quad + \int\limits_I v^{(2)}_{fu}(x,u,f)h_1R(h_2)dx + \int\limits_I v^{(2)}_{ff}(x,u,f)h_1h_2dx. \qquad (2.4)
\end{aligned}
$$

The linearity and bilinearity of $G^{(1)}(f)$ and $G^{(2)}(f)$ are evidents.
Let us prove now that they are bounded.

$$
\begin{aligned}
|G^{(1)}(f)h| &\leq \int\limits_I |v^{(1)}_u(x,u,f)||R(h)|dx + \int\limits_I |v^{(1)}_f(x,u,f)||h|dx \\
&\leq \int\limits_I \Big[a\Big(|u(x)|^{p+2}+|f(x)|^{p+2}\Big)+|b_5(x)|\Big]\Big[|R(h)|+|h)|\Big]dx \\
&\leq \int\limits_I \Big[a\Big(|u(x)|^{p+2}+|f(x)|^{p+2}\Big)+|b_5(x)|\Big]\Big[\max_{x\in\bar I}(R(h))(x)+\max_{x\in\bar I}h(x)\Big]dx \\
&= \Big[a\Big(\|u(x)\|^{p+2}_{L_{p+2}(I)}+\|f(x)\|^{p+2}_{L_{p+2}(I)}\Big)+\|b_5(x)\|_{L_1(I)}\Big]\Big[\|R(h)\|_{C(\bar I)}+\|h\|_{C(\bar I)}\Big] \\
&\leq c_2\Big[\|h\|_{W^1_2(I)}+\|R(h)\|_{W^1_2(I)}\Big].
\end{aligned}
$$

Thus $\exists\ c_2>0$ such that $|G^{(1)}(f)h| \leq c_2\Big(\|R(h)\|_{W^1_2(I)}+\|h\|_{W^1_2(I)}\Big)$.
Since $R(h)$ depends continually of h, then $|G^{(1)}(f)h|\leq c_3\|h\|_{W^1_2(I)}$,
where $c_3>0$.
Consequently $G^{(1)}(f)\in\mathcal{L}((W^{1,2}_0(I),\|.\|_{W^1_2(I)}),\mathbb{R})$.

$$
|G^{(2)}(f)(h_1,h_2)| \leq \int\limits_I\Big[a\Big(|u|^p+|f|^p\Big)+|b_6(x)|\Big]
$$

$$\Big[|R(h_1)||R(h_2)| + |R(h_1)||h_2| + |h_1||R(h_2)| + |h_1||h_2|\Big]dx$$

$$\begin{aligned}
&\leq \Big[a\Big(\|u(x)\|^p_{L_p(I)} + \|f(x)\|^p_{L_p(I)}\Big) + \|b_6(x)\|_{L_1(I)}\Big] \\
&\quad \Big[\max_{x\in\bar I}|[R(h_1)](x)| \max_{x\in\bar I}|[R(h_2)](x)| + \max_{x\in\bar I}|[R(h_1)](x)| \max_{x\in\bar I}|h_2(x)| \\
&\quad + \max_{x\in\bar I}|h_1(x)| \max_{x\in\bar I}|[R(h_2)](x)| + \max_{x\in\bar I}|h_1(x)| \max_{x\in\bar I}|h_2(x)|\Big] \\
&\leq c_3\Big[\|u\|_{W_2^1(I)} + \|f\|_{W_2^1(I)} + \|b_6(x)\|_{L_1(I)}\Big] \\
&\quad \Big[\|R(h_1)\|_{C(\bar I)}\|R(h_2)\|_{C(\bar I)} + \|R(h_1)\|_{C(\bar I)}\|h_2\|_{C(\bar I)} \\
&\quad + \|h_1\|_{C(\bar I)}\|R(h_2)\|_{C(\bar I)} + \|h_1\|_{C(\bar I)}\|h_2\|_{C(\bar I)}\Big] \\
&\leq c_4\Big[\|R(h_1)\|_{W_2^1(I)}\|R(h_2)\|_{W_2^1(I)} + \|R(h_1)\|_{W_2^1(I)}\|h_2\|_{W_2^1(I)} \\
&\quad + \|h_1\|_{W_2^1(I)}\|R(h_2)\|_{W_2^1(I)} + \|h_1\|_{W_2^1(I)}\|h_2\|_{W_2^1(I)}\Big] \\
&\leq c_5\|h_1\|_{W_2^1(I)}\|h_2\|_{W_2^1(I)}.
\end{aligned}$$

Thus $G^{(2)}(f) \in \mathcal{B}((W_0^{1,2}(I), \|.\|_{W_2^1(I)}), \mathbb{R})$.
Let us prove now that G is $(\tau_p, \|.\|_{W_2^1(I)})$ Taylor mapping, where τ_p is a topology generated by $L_p(I)$.
Let $f \in F$, and let us prove that :

$$r(h) \equiv G(f+h) - G(f) - G^{(1)}(f)h - 2^{-1}G^{(2)}(f)(h,h)$$

is $(\tau_p, \|.\|_{W_2^1(I)})$ infinitely small of second order at zero.
Assume the contrary, then $\exists(\tilde h_m)_{m\in\mathbb{N}} \in F$ and $\varepsilon > 0$ such that
$\tilde h_m \to 0$ in $L_p(I)$ and $r(\tilde h_m) \geq \varepsilon\|\tilde h_m\|^2_{W_2^1(I)}$.
Using the Agmon's-Douglis-Niremberg's theorem, we obtain
$R(\tilde h_m) \to 0$ in $L_p(I)$, and using the lemma (1.1), we deduce that there exists
$\tilde Z(x)$ in $L_p(I)$: $|(R(\tilde h_m))(x)| \leq \tilde z(x)$.
let $\tilde Z_0(x) = \tilde z(x) + |u(x)|$, then $|u(x)| + |(R(\tilde h_m))(x)| \leq \tilde Z_0(x)$ and $Z_0 \in L_p(I)$.
Analogously, for $f \in W_2^1(I)$, we obtain $|f(x)| + |\tilde h_m| \leq \tilde Z_1$, where $Z_1 \in L_p(I)$ We have

$$r(h) \;=\; \int_I \Big[v(x, u+R(h), f+h) - v(x,u,f) - v_u^{(1)}(x,u,f)R(h) - v_f^{(1)}(x,u,f)h$$

$$-2^{-1}\Big[v^{(2)}_{uu}(x,u,f)R^2(h)+2v^{(2)}_{uf}(x,u,f)R(h)h+v^{(2)}_{ff}(x,u,f)h^2\Big]\Big]dx.$$

And since

$$\begin{aligned}
v(x,u+R(h),f+h)-v(x,u,f) &= v(x,u+R(h),f+h)-v(x,u,f+h)\\
&\quad +v(x,u,f+h)-v(x,u,f)\\
&= \int_0^1 v^{(1)}_u(x,u+\theta R(h),f+h)R(h)d\theta\\
&\quad +\int_0^1 v^{(1)}_f(x,u,f+\lambda h)hd\lambda\\
&= \int_0^1 R(h)\Big[v^{(1)}_u(x,u+\theta R(h),f+h)\\
&\quad -v^{(1)}_u(x,u+\theta R(h),f)+v^{(1)}_u(x,u+\theta R(h),f)\Big]d\theta\\
&\quad +\int_0^1 v^{(1)}_f(x,u,f+\lambda h)hd\lambda\\
&= \int_0^1 R(h)v^{(1)}_u(x,u+\theta R(h),f)d\theta\\
&\quad +\int_0^1 v^{(1)}_f(x,u,f+\lambda h)hd\lambda\\
&\quad +\int_0^1\int_0^1 v^{(2)}_{fu}(x,u+\theta R(h),f+\lambda h)hR(h)d\lambda d\theta,
\end{aligned}$$

then

$$\begin{aligned}
r(h) &= \int_I\int_0^1\Big[v^{(1)}_u(x,u+\theta R(h),f+h)R(h)-v^{(1)}_u(x,u,f)R(h)-2^{-1}v^{(2)}_{uu}(x,u,f)R^2(h)\Big]d\theta dx\\
&\quad +\int_I\int_0^1\Big[v^{(1)}_f(x,u,f+\lambda h)h-v^{(1)}_f(x,u,f)h-2^{-1}v^{(2)}_{ff}(x,u,f)h^2\Big]d\lambda dx\\
&\quad -\int_I\int_0^1 v^{(2)}_{uf}(x,u,f)R(h)hd\lambda dx+\int_I\int_0^1\int_0^1 v^{(2)}_{uf}(x,u+\theta R(h),f+\lambda h)hR(h)d\lambda d\theta dx\\
&= \int_I\int_0^1\Big[v^{(1)}_u(x,u+\theta R(h),f)-v^{(1)}_u(x,u,f)-\theta v^{(2)}_{uu}(x,u,f)R(h)\Big]R(h)d\theta dx\\
&\quad +\int_I\int_0^1\Big[v^{(1)}_f(x,u,f+\lambda h)-v^{(1)}_f(x,u,f)-\lambda v^{(2)}_{ff}(x,u,f)h\Big]hd\lambda dx
\end{aligned}$$

$$-\int\limits_I\int_0^1 v_{uf}^{(2)}(x,u,f)R(h)hd\lambda dx+\int\limits_I\int_0^1\int_0^1 v_{uf}^{(2)}(x,u+\theta R(h),f+\lambda h)hR(h)d\lambda d\theta dx.$$

let A_m, B_m two functions defined by :

$$A_m(x,\theta)=\begin{cases}\dfrac{v_u^{(1)}(x,u+\theta R(\tilde{h}_m),f)-v_u^{(1)}(x,u,f)}{R(\tilde{h}_m)}-\theta v_{uu}^{(2)}(x,u,f) & \text{if } R(\tilde{h}_m)\neq 0\\ 0 & \text{if } R(\tilde{h}_m)=0\end{cases}$$

$$B_m(x,\lambda)=\begin{cases}\dfrac{v_f^{(1)}(x,u,f+\lambda\tilde{h}_m)-v_f^{(1)}(x,u,f)}{\tilde{h}_m}-\lambda v_{ff}^{(2)}(x,u,f) & \text{if } \tilde{h}_m\neq 0\\ 0 & \text{if } \tilde{h}_m=0\end{cases}$$

Let F_m defined by:
$F_m(x,\theta,\lambda)=v_{uf}^{(2)}(x,u(x)+\theta R(\tilde{h}_m),f+\lambda\tilde{h}_m)-v_{uf}^{(2)}(x,u(x),f)$,
then

$$\begin{aligned} r(\tilde{h}_m)| &= \Big|\int\limits_I\int_0^1 A_m(x,\theta)R^2(\tilde{h}_m)d\theta dx+\int\limits_I\int_0^1 B_m(x,\lambda)\tilde{h}_m^2 d\lambda dx\\ &\quad+\int\limits_I\int_0^1\int_0^1 F_m(x,\theta,\lambda)R(\tilde{h}_m)\tilde{h}_m d\lambda d\theta dx\Big|.\end{aligned}$$

So we obtain

$$\begin{aligned} |r(\tilde{h_m})| &\leq \int_0^1\int\limits_I |A_m(x,\theta)|dxd\theta\max_{x\in\bar{I}}|[R(\tilde{h}_m)](x)|^2\\ &\quad+\int_0^1\int\limits_I |B_m(x,\lambda)|dxd\lambda\max_{x\in\bar{I}}|\tilde{h}_m|^2\\ &\quad+\int_0^1\int_0^1\int\limits_I |F_m(x,\theta,\lambda)|dxd\lambda d\theta\max_{x\in\bar{I}}|[R(\tilde{h}_m)](x)|\max_{x\in\bar{I}}|\tilde{h}_m|\\ &\leq c_6\Big[\int_0^1\int\limits_I |A_m(x,\theta)|dxd\theta+\int_0^1\int\limits_I |B_m(x,\lambda)|dxd\lambda\\ &\quad+\int_0^1\int_0^1\int\limits_I |F_m(x,\theta,\lambda)|dxd\lambda d\theta\Big]\|\tilde{h}_m)\|_{W_2^1(I)}, \qquad (2.8)\end{aligned}$$

and

$$|A_m(x,\theta)| \leq |v_{uu}^{(2)}(x,u(x)+k_m(x)\theta[R(\tilde{h_m})](x),f)|+|v_{uu}^{(2)}(x,u(x),f)|$$

$$\leq a\Big[\Big(|u(x)|+|R(\tilde{h_m})|\Big)^p+|f(x)|^p\Big]+|b_6(x)|+a\Big[|u(x)|^p+|f(x)|^p\Big]+|b_6(x)|$$

$$\leq a\Big[|\tilde{Z}_0(x)|^p+2|f(x)|^p+|u(x)|^p\Big]+2|b_6(x)|\in L_1(I).$$

Analogously

$$\begin{aligned}|B_m(x,\lambda)| &\leq |v_{ff}^{(2)}(x,u(x),\lambda S_m(x)\tilde{h}_m(x)+f)|+|v_{ff}^{(2)}(x,u(x),f)|\\ &\leq a\Big[|u(x)|^p+\Big(|f(x)|+|\tilde{h}_m|\Big)^p\Big]+|b_6(x)|+a\Big[|u(x)|^p+|f(x)|^p\Big]+|b_6(x)|\\ &\leq a\Big[2|u(x)|^p+|\tilde{Z}_1(x)|^p+|f(x)|^p\Big]+2|b_6(x)|\in L_1(I).\end{aligned}$$

$$\begin{aligned}|F_m(x,\theta,\lambda)| &\leq |v_{uf}^{(2)}(x,u(x),f)|+|v_{uf}^{(2)}(x,u(x)+\theta R(\tilde{h}_m),f+\lambda\tilde{h}_m)|\\ &\leq a\Big[|u(x)|^p+|f(x)|^p\Big]+|b_6(x)|+a\Big[|\tilde{Z}_0(x)|^p+|\tilde{Z}_1(x)|^p\Big]+|b_6(x)| \quad \in L_1(I).\end{aligned}$$

Or $A_m(x,\theta)\to 0, B_m(x,\lambda)\to 0, F_m(x,\theta,\lambda)\to 0$ almost everywhere,then

$$\int_0^1\int_I |A_m(x,\theta)|dxd\theta\to 0\ \ m\to+\infty$$

$$\int_0^1\int_I |B_m(x,\lambda)|dxd\lambda\to 0\ \ ,m\to+\infty$$

$$\int_0^1\int_0^1\int_I |F_m(x,\theta,\lambda)|dxd\lambda d\theta\to 0\ \ ,m\to+\infty$$

. which contradicts the assumption (2.8).
Thus we acheve the Proof.

Theorem 2.2 *Suppose that in addition of conditions of theorem (2.1) we have:*

$$|v_u^{(1)}(x,u,f)|+|v_f^{(1)}(x,u,f)| \ \leq\ a\Big(|u|^p+|f|^p\Big)+|\widehat{b}_3(x)|,$$

where $\widehat{b}_3(x)\in L_1(I)$.
Then the functional G *is* $(\tau_p,\|.\|_{W_2^1(I)})$ *Taylor mapping of first and second order at every* $f\in F$,
and $G^{(1)}(f)\in\mathcal{L}((W_0^{1,2}(I),\|.\|_{W_2^1(I)}),\mathbb{R})$.

Proof

Let $f \in F$, and let us prove that the map

$r(h) \equiv G(f+h) - G(f) - G^{(1)}(f)h$ is $(\tau_p, \|.\|_{W_2^1(I)})$ infinitely small of first order at zero.

Assume the contrary, then

$\exists\ \tilde{h}_m \in L_p(I)$ such that $\tilde{h}_m \to 0$ in $L_p(I)$ but $|r(\tilde{h}_m)| \geq \varepsilon \|\tilde{h}_m\|_{W_2^1(I)}$.

Thus $R(\tilde{h}_m) \to 0$ in $L_p(I)$, as $m \to +\infty$.

Using the lemma (1.1),we deduce that there exists $\tilde{Z}_0(x)$ in $L_p(I)$ and there exists $\tilde{Z}_1(x)$ in $L_p(I)$: forall m in $\mathbb{N}$ $|u(x)| + |R(\tilde{h}_m)| \leq \tilde{Z}_0(x)$ almost everywhere in I.

We have also forall m in $\mathbb{N}$ $|f(x)| + |\tilde{h}_m| \leq \tilde{Z}_1(x)$ almost everywhere in I.

So,

$$\begin{aligned}
|r(\tilde{h}_m)| &= \Big| \int_I \Big[v(x, u(x) + [R(\tilde{h}_m)](x), f(x) + \tilde{h}_m(x)) - v(x, u(x), f(x)) \\
&\quad - v_u^{(1)}(x, u(x), f(x)) R(\tilde{h}_m) - v_f^{(1)}(x, u(x), f(x)) \tilde{h}_m \Big] dx \Big| \\
&\leq \Big| \int_I \Big[v(x, u(x) + R(\tilde{h}_m), f(x) + \tilde{h}_m) - v(x, u(x), f(x) + \tilde{h}_m) \\
&\quad - v_u^{(1)}(x, u(x), f(x)) R(\tilde{h}_m) + v(x, u(x), f(x) + \tilde{h}_m) \\
&\quad - v(x, u(x), f(x)) - v_f^{(1)}(x, u(x), f(x)) \tilde{h}_m \Big] dx \Big| \\
&= \Big| \int_I A_m(x) R(\tilde{h}_m) dx \Big| + \Big| \int_Q B_m(x) \tilde{h}_m dx \Big|,
\end{aligned}$$

where,

$$A_m(x) = \begin{cases} \dfrac{v(x, u + R(\tilde{h}_m), f + \tilde{h}_m) - v(x, u, f + \tilde{h}_m)}{R(\tilde{h}_m)} - v_u^{(1)}(x, u, f) & \text{if } R(\tilde{h}_m) \neq 0 \\ 0 & \text{if } R(\tilde{h}_m) = 0 \end{cases}$$

$$B_m(x) = \begin{cases} \dfrac{v(x, u, f + \tilde{h}_m) - v(x, u, f)}{\tilde{h}_m} - v_f^{(1)}(x, u, f) & \text{if } \tilde{h}_m \neq 0 \\ 0 & \text{if } \tilde{h}_m = 0 \end{cases}.$$

We have

$$|r(\tilde{h}_m)| \leq \int_I |A_m(x)| dx \max_{x \in \bar{I}} \Big| [R(\tilde{h}_m)](x) \Big| + \int_I |B_m(x)| dx \max_{x \in \bar{I}} \Big| \tilde{h}_m(x) \Big|$$

$$\leq c_3\Big[\int_I |A_m(x)|dx + \int_I |B_m(x)|dx\Big]\|\tilde{h}_m\|_{W_2^1(I)}. \qquad (2.9)$$

let us remark that: $A_m(x) \to 0, B_m(x) \to 0$ almost everywhere in I.
Using the mean value theorem , we obtain

$$\begin{aligned}
|A_m(x)| &\leq a\Bigg[\Big(\Big|u\Big| + \Big|R(\tilde{h}_m)(x)\Big|\Big)^p + \Big(\Big|f(x)\Big| + \Big|\tilde{h}_m(x)\Big|\Big)^p\Bigg] + |\widehat{b}_3(x)| \\
&+a\Bigg[|u(x)|^p + |f(x)|^p\Bigg] + |\widehat{b}_3(x)| \\
&\leq a\Bigg[|\tilde{Z}_0(x)|^p + |\tilde{Z}_1(x)|^p\Bigg] + 2|\widehat{b}_3(x)| + a\Bigg[|u(x)|^p + |f(x)|^p\Bigg] \in L_1(I).
\end{aligned}$$

and

$$\begin{aligned}
|B_m(x)| &\leq a\Bigg[|u(x)|^p + \Big(\Big|f(x)\Big| + \Big|\tilde{h}_m(x)\Big|\Big)^p\Bigg] + |\widehat{b}_3(x)| \\
&+a\Bigg[|u(x)|^p + |f(x)|^p\Bigg] + |\widehat{b}_3(x)| \\
&\leq a\Bigg[2|u(x)|^p + |\tilde{Z}_1(x)|^p\Bigg] + a|f(x)|^p + 2|\widehat{b}_3(x)| \in L_1(I).
\end{aligned}$$

Using the dominated convergence theorem , we conclude that

$$\int_I |A_m(x)|dx \to 0, \text{as} \quad m \to +\infty$$

and

$$\int_I |B_m(x)|dx \to 0, \text{as} \quad m \to +\infty$$

which contradicts the assumption (2.9).
Thus we acheve the proof.
Now let us giving the sufficient conditions of optimality for the problem (1.3),(1.4)and(1.5).

Theorem 2.3 *Suppose that for the problem (1.5),v_k satisfies the conditions of theorem (2.1) and (2.2) then the functionals*

$$J_k(f) \equiv \int_I v_k(x,u,f)dx \quad (k = s_1+1, ..., s_1+s_2)$$

are $(\tau_p, \|.\|_{W_2^1(I)})$ *Taylor mapping of first and second order at every* $f \in F$ *and* $J_k(f) = \int\limits_I v_k(x,u,f)dx + c_k\|f\|^2_{W_2^1(I)}$ $(k = 0,..,s_1)$ *are lower semi Taylor mapping of first and second order at every* $f \in F$.
Consequently,
$\exists J_k^{(1)}(f)$, $\exists J_k^{(2)}(f)$ $(k = 0,..,s_1+s_2)$.
Let us suppose also that $\widehat{f} \in F, J(\widehat{f}) = 0,\ J_k(\widehat{f}) = 0$
and put $L = \{h \in W_0^{1,2}(I)/J_k^{(1)}(\widehat{f})h = 0, k = 1,..,s_1, J^{(1)}(\widehat{f})h = 0\}$.
Suppose that $\exists\widehat{\lambda} \in (\mathbb{R}^{s_1})^*, \exists\widehat{y}^* \in (\mathbb{R}^{s_2})^*, \exists\gamma \geq 0,\ \exists\widehat{\lambda}_k > 0 (k = 1,..,s_1)$:
$\mathcal{L}_f(\widehat{f},\widehat{y}^*,\widehat{\lambda},1) = 0$ *and* $\forall h \in L \quad \mathcal{L}_{ff}(\widehat{f},\widehat{y}^*,\widehat{\lambda},1)(h,h) \geq 2\gamma\|h\|^2_{W_2^1(I)}$,
where $\mathcal{L}_f(\widehat{f},\widehat{y}^*,\widehat{\lambda},1)$ *and* $\mathcal{L}_{ff}(\widehat{f},\widehat{y}^*,\widehat{\lambda},1)$ *are defined by formula (1.7) and (1.8).*
Then $\widehat{f}$ *is* τ_p *strict local minimum.*

Proof

All conditions of theorem (1.6) in [1] are satisfied, so $\widehat{f}$ is a strict τ_p-local minimum point.

Theorem 2.4 *suppose that in the problem (1.3),* v_k *satisfies the conditions of theorem (2.2) and (2.3) then the functionals*

$$J_k(f) \equiv \int\limits_I v_k(x,u,f)dx \quad (k = s_1+1,..,s_1+s_2)$$

are $(\tau_p, \|.\|_{W_2^1(I)})$ *Taylor mapping of first and second order at every* $f \in F$ *and* $J_k(f) = \int\limits_I v_k(x,u,f)dx + c_k\|f\|^2_{W_2^1(I)}$ $(k = 0,..,s_1)$ *are lower semi Taylor mapping of first and second order at every* $f \in F$.
Consequently, $\exists J_k^{(1)}(f)$ *and* $\exists J_k^{(2)}(f)$ $(k = 0,..,s_1+s_2)$
Let us suppose also that
$J_0^{(1)}(\widehat{f}) = 0$ *and* $\exists\alpha > 0, \quad \forall h \in W_0^{1,2}(I) \quad J_0^{(2)}(\widehat{f})(h,h) \geq 2\alpha\|h\|^2_{W_2^1(I)}$.
Then $\widehat{f}$ *is* τ_p *strict local minimum .*

Proof

All conditions of theorem (1.4) in [1] are satisfied, so $\widehat{f}$ is a strict τ_p-local minimum point.

Theorem 2.5 *Suppose that in the problem (1.4),v_k satisfies the conditions of theorem (2.1) and (2.2) then the functionals*

$$J_k(f) \equiv \int_I v_k(x,u,f)dx \quad k = s_1+1,..,s_1+s_2$$

are $(\tau_p, \|.\|_{W_2^1(I)})$ *Taylor mapping of first and second order at every* $f \in F$ *and* $J_k(f) = \int_I v_k(x,u,f)dx + c_k\|f\|^2_{W_2^1(I)}$ *are lower semi Taylor mapping of first and second order at every* $f \in F \;\; \forall\; k = 0,..,s_1$.
Consequently, $\exists J_k^{(1)}(f)$ *and* $\exists J_k^{(2)}(f)$ $(k = 0,..,s_1+s_2)$
Let us suppose also that $J(\widehat{f}) = 0$, $\exists\; \widehat{y}^* \in (\mathbb{R}^{s_2})^*$, $\exists \alpha > 0 \;\; \mathcal{L}_f(\widehat{f}, \widehat{y}^*, 1) = 0$
and $\forall h \in \; kerJ^{(1)}(\widehat{f}) \;\; \mathcal{L}_{ff}(\widehat{f}, \widehat{y}^*, 1)(h,h) \geq 2\alpha\|h\|^2_{W_2^1(I)}$,
where $\mathcal{L}_f(\widehat{f}, \widehat{y}^*, 1)$ *and* $\mathcal{L}_{ff}(\widehat{f}, \widehat{y}^*, 1)$ *are given by formula (1.10),(1.11).*
Then $\widehat{f}$ *is* τ_p *strict local minimum.*

Proof

All conditions of theorem (1.5) in [1] are satisfied, so $\widehat{f}$ is a strict τ_p-local minimum point.

Remark 2.1 *Let us remark that in the theorem (2.1) and (2.2),the increasing conditions verified by v are not sufficient to certify the Frechet differentiability of functional* $G : (W_0^{1,2}(I), \|.\|_{L_p(I)}) \to \mathbb{R}^{s_2}$.

In fact,
suppose we have $\frac{3}{2} < p < 2$. Let us define $v : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by: $v(x,u,f) = a\Big[|u|^{\frac{5}{2}} + |f|^{\frac{5}{2}}\Big] + |b_0(x)|$,
where $b_0(x) \in C(\overline{I}), a \in \mathbb{R}, a > 0$.
Let $d_m \to +\infty$, and put $\alpha_m = |d_m|^{\frac{1}{2}}$,then $\alpha_m \to +\infty$
and $\forall x \in I \;\; \forall u \in \mathbb{R} \;\; \forall m \in \mathbb{N}$ We have

$$\begin{aligned} |v(x,u,d_m)| &\geq a|d_m|^{\frac{5}{2}} \\ &= a|d_m|^{\frac{1}{2}}|d_m|^2 \\ &\geq a|d_m|^{\frac{1}{2}}|d_m|^p \\ &= a\alpha_m|d_m|^p. \end{aligned}$$

Let $\tilde{f} \in W_0^{1,2}(I)$.
By the countable additivity of lebesgue measure, $\exists c > 0,\ \ \exists I' \subset I:\quad \mu(I') > 0,\ \ \rho(I', \partial I) > 0$ and $\forall x \in I'\ \ |\tilde{f}(x)| \leq c$.
Put $\mathcal{D} \equiv \max\{|v(x,u,f)|/|u| \leq c, |f| \leq c, x \in \overline{I}\} < \infty$.
Let us choose $I_m \subset I'$ such that $\mu(I_m) = |d_m|^{-p} \alpha_m^{-\frac{1}{2}}$.
Let $\tilde{h}_m$ defined by:

$$\tilde{h}_m(x) = \begin{cases} d_m - \tilde{f}(x) & \text{if } x \in I_m \\ 0 & \text{if } x \in I \setminus I_m \end{cases}$$

We have

$$\begin{aligned} \|\tilde{h}_m(x)\|_{L_p(I)} &\leq \|d_m\|_{L_p(I_m)} + |\tilde{f}(x)\|_{L_p(I_m)} \\ &\leq d_m \Big(\mu(I_m)\Big)^{\frac{1}{p}} + c\Big(\mu(I_m)\Big)^{\frac{1}{p}} \\ &\leq \alpha_m^{-\frac{1}{2p}} + c\Big(\mu(I_m)\Big)^{\frac{1}{p}}. \end{aligned}$$

Consequently $\|\tilde{h}_m(x)\|_{L_p(I)} \to 0$, i.e $\tilde{h}_m(x) \to 0$ in $L_p(I)$.
We have also

$$\begin{aligned} |G(\tilde{f} + \tilde{h}_m) - G(\tilde{f})| &= \Bigg| \int\limits_{I_m} \Big[v(x, [R(\tilde{f} + \tilde{h}_m)](x), \tilde{f}(x) + \tilde{h}_m(x)) \\ &\quad - v(x, [R(\tilde{f})](x), \tilde{f}(x)) \Big] dx \Bigg| \\ &\geq \Bigg| \int\limits_{I_m} v(x, [R(\tilde{f} + \tilde{h}_m)](x), d_m) dx \Bigg| \\ &\quad - \Bigg| \int\limits_{I_m} v(x, [R(\tilde{f})](x), \tilde{f}(x)) dx \Bigg| \\ &\geq a\alpha_m |d_m|^p - \mathcal{D}\mu(I_m) \\ &= a\alpha_m^{\frac{1}{2}} \to +\infty. \end{aligned}$$

Therfore, $|G(\tilde{f} + \tilde{h}_m) - G(\tilde{f})| \to +\infty$.
Thus G is not Frechet differentiable at every $f \in W_0^{1,2}(I)$.

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